Preview of Calculus

Site:	Saylor Academy	Printed by:	Guest user
Course:	MA005: Calculus I	Date:	Monday, September 30, 2024, 2:12 AM
Book:	Preview of Calculus		

Description

Read this section for an introduction to calculus.

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Two Basic Problems

Beginning calculus can be thought of as an attempt, a historically successful attempt, to solve two fundamental problems. In this section we will start to examine geometric forms of those two problems and some fairly simple ways to attempt to solve them. At first, the problems themselves may not appear very interesting or useful, and the methods for solving them may seem crude, but these simple problems and methods have led to one of the most beautiful, powerful, and useful creations in mathematics: Calculus.

First Problem: Finding the Slope of a Tangent Line

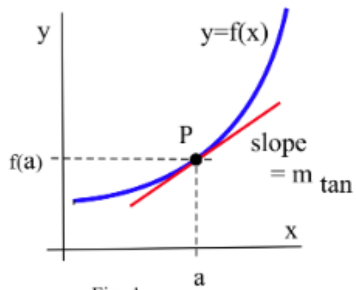


Fig. 1

Suppose we have the graph of a function $y = f(x)$, and we want to find the equation of the line which is **tangent** to the graph at a particular point P on the graph (Fig. 1). (We will give a precise definition of tangent in Section 1.0. For now, think of the **tangent** line as the line which touches the curve at the point P and stays close to the graph of $y = f(x)$ near P). We know that the point P is on the tangent line, so if the x–coordinate of P is $x = a$, then the y–coordinate of P must be $y = f(a)$ and $P = (a, f(a))$. The only other information we need to find the equation of the tangent line is its slope, m_{tan}, and that is where the difficulty arises. In algebra, we needed two points in order to determine a slope, and so far we only have the point P. Lets simply pick a second point, say Q, on the graph of $y = f(x)$. If the x–coordinate of Q is b (Fig. 2), then the y–coordinate is $f(b)$, so $Q = (b, f(b))$. The slope of the line through P and Q is

$$m_{PQ} = \frac{rise}{run} = \frac{f(b) – f(a)}{b – a}$$

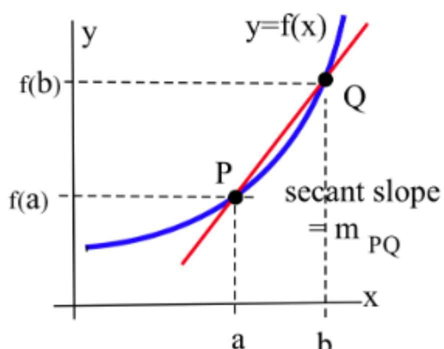


Fig. 2

If we drew the graph of $y = f(x)$ on a wall, put nails at the points P and Q on the graph, and laid a straightedge on the nails, then the straightedge would have slope m_{PQ} (Fig. 2). However, the slope m_{PQ} can be very different from the value we want, the slope m_{tan} of the tangent line. The key idea is that if the point Q is close to the point P, then the slope m_{PQ} is close to the slope we want, m_{tan}. Physically, if we slide the nail at Q along the graph towards the fixed point P, then the slope, $m_{PQ} = \dfrac{f(b) – f(a)}{b – a}$, of the straightedge gets closer and closer to the slope, m_{tan} , of the tangent line. If the value of b is very close to a, then the point Q is very close to P, and the value of m_{PQ} is very close to the value of m_{tan}. Rather than defacing walls with graphs and nails, we can calculate

$$m_{PQ} = \frac{f(b) – f(a)}{b – a}$$

and examine the values of m_{PQ} as b gets closer and closer to a. We say that m_{tan} is the limiting value of m_{PQ} as b gets very close to a, and we write

$$\mathrm{m_{tan}} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

The slope mtan of the tangent line is called the **derivative** of the function $f(x)$ at the point P, and this part of calculus is called **differential calculus**. Chapters 2 and 3 begin differential calculus.

The slope of the tangent line to the graph of a function will tell us important information about the function and will allow us to solve problems such as:

"The US Post Office requires that the length plus the girth (Fig. 3) of a package not exceed 84 inches. What is the largest volume which can be mailed in a rectangular box?"

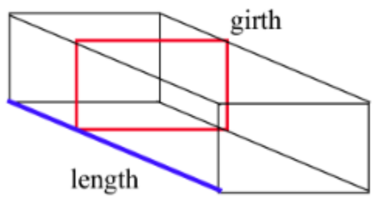


Fig. 3

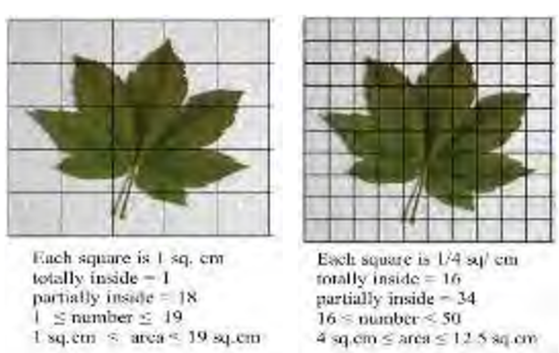
An oil tanker was leaking oil, and a 4 inch thick oil slick had formed. When first measured, the slick had a radius 200 feet and the radius was increasing at a rate of 3 feet per hour. At that time, how fast was the oil leaking from the tanker?

Derivatives will even help us solve such "traditional" mathematical problems as finding solutions of equations like $x^2 = 2 + sin(x)$ and $x^9 + 5x^5 + x^3 + 3 = 0$.

Second Problem: Finding the Area of a Shape

Suppose we need to find the area of a leaf (Fig. 4) as part of a study of how much energy a plant gets from sunlight. One method for finding the area would be to trace the shape of the leaf onto a piece of paper and then divide the region into "easy" shapes such as rectangles and triangles whose areas we could calculate.

Fig. 4



We could add all of the "easy" areas together to get the area of the leaf. A modification of this method would be to trace the shape onto a piece of graph paper and then count the number of squares completely inside the edge of the leaf to get a lower estimate of the area and count the number of squares that touch the leaf to get an upper estimate of the area. If we repeat this process with smaller squares, we have to do more counting and adding, but our estimates are closer together and closer to the actual area of the leaf. (This area can also be approximated using a sheet of paper, scissors and an accurate scale. How?)

We can calculate the area A between the graph of a function $y = f(x)$ and the x–axis (Fig. 5) by using similar methods. We can divide the area into strips of width w and determine the lower and upper values of $y = f(x)$ on each strip. Then we can approximate the area of each rectangle and add all of the little areas together to get A_w, an approximation of the exact area. The key idea is that if w is small, then the rectangles are narrow, and the approximate area Aw is very close to the actual area A. If we take narrower and narrower rectangles, the approximate areas get closer and closer to the actual area: $\mathrm{A} = limit_{w \to 0} A_w$.

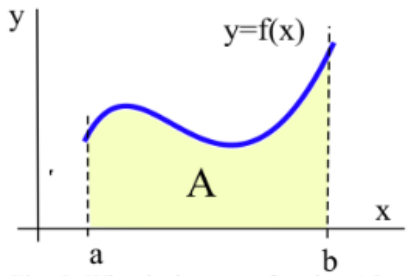


Fig. 5: What is the area of region A?

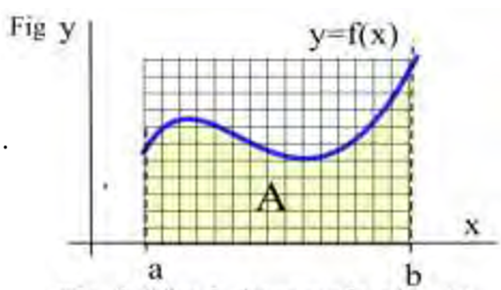


Fig. 5: What is the area of region A?

The process we used is the basis for a technique called **integration**, and this part of calculus is called **integral calculus**. Integral calculus and integration will begin in Chapter 4.

The process of taking the limit of a sum of "little" quantities will give us important information about the function and will also allow us to solve problems such as:

"Find the length of the graph of $y = sin(x)$ over one period (from $x = 0$ to $x = 2\pi$)".

"Find the volume of a torus ("doughnut") of radius 1 inch which has a hole of radius 2 inches. (Fig. 6)"

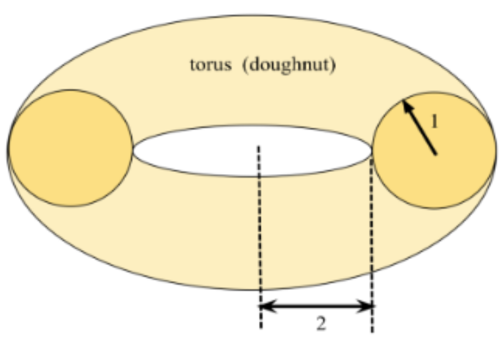


Fig.6: What is the volume of the torus?

"A car starts at rest and has an acceleration of $5 + 3sin(t)$ feet per second per second in the northerly direction at time t seconds. Where will the car be, relative to its starting position, after 100 seconds?"

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A Unifying Process: Limits

We used a similar processes to "solve" both the tangent line problem and the area problem. First, we found a way to get an approximate solution, and then we found a way to improve our approximation. Finally, we asked what would happen if we continued improving our approximations "forever", that is, we "took a limit". For the tangent line problem, we let the point Q get closer and closer and closer to P, the limit as b approached a. In the area problem, we let the widths of the rectangles get smaller and smaller, the limit as w approached 0. Limiting processes underlie derivatives, integrals, and several other fundamental topics in calculus, and we will examine limits and their properties in Chapter 1.

Two Sides Of The Same Coin: Differentiation and Integration

Just as the set–up of each of the two basic problems involved a limiting process, the solutions to the two problems are also related. The process of differentiation for solving the tangent line problem and the process of integration for solving the area problem turn out to be "opposites" of each other: each process undoes the effect of the other process.

The Fundamental Theorem of Calculus in Chapter 4 will show how this "opposite" effect works.

Problem for Solution

The first 5 chapters present the two key ideas of calculus, show "easy" ways to calculate derivatives and integrals, and examine some of their applications. And there is more. In later chapters, new functions will be examined and ways to calculate their derivatives and integrals will be found. The approximation ideas will be extended to use "easy" functions, such as polynomials, to approximate the values of "hard" functions such as $sin(x)$ and e^x . And the notions of "tangent lines" and "areas" will be extended to 3–dimensional space as "tangent planes" and "volumes".

Success in calculus will require time and effort on your part, but such a beautiful and powerful field is worth that time and effort.